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Solution

Question 1

1 The moment generating function of a random variable Y is given by

$$M_Y(t) = \left(\frac{1}{1-t}\right)^2$$

From the table of moment generating function in the Probability Text Book by *Ross*, we know that gamma distribution with parameter $(2,1)$ will have a moment generating function of $M_Y(t)$.

The probability density function of gamma distribution with parameter $(2,1)$ is given by

$$f_Y(y) = \begin{cases} \dfrac{e^{-y}y^1}{\Gamma(2)} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Then, we can compute $P(-2 < Y < 3)$ from its density function as follow:

$$\begin{aligned} P(-2 < Y < 3) &= \int_{-2}^{3} f_Y(y)dy \\ &= \int_{-2}^{0} 0dy + \int_{0}^{3} \frac{e^{-y}y^1}{\Gamma(2)} dy \\ &= 0 + \int_{0}^{3} ye^{-y}dy \\ &= [-ye^{-y}]_0^3 - \int_0^3 -e^{-y}dy \\ &= -3e^{-3} - [e^{-y}]_0^3 \\ &= -3e^{-3} - e^{-3} + e^0 \\ &= 1 - 4e^{-3} \approx 0.8009 \end{aligned}$$

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Question 2

2 By Markov inequality,

$$P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right| > \epsilon\right) = P\left(\left(\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right)^2 > \epsilon^2\right)$$

$$\leq \frac{E\left[\left(\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right)^2\right]}{\epsilon^2} = \frac{Var\left(\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right) + \left[E\left(\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right)\right]^2}{\epsilon^2}$$

$$= \frac{\frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) + \left[\frac{1}{n} E(\sum_{i=1}^{n} X_i) - \mu\right]^2}{\epsilon^2}$$

$$= \frac{\frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 + \left[\frac{1}{n} \sum_{i=1}^{n} \mu_i - \mu\right]^2}{\epsilon^2}$$

Taking limits to infinity, we have

$$\lim_{n\to\infty} \frac{\frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 + \left[\frac{1}{n} \sum_{i=1}^{n} \mu_i - \mu\right]^2}{\epsilon^2} = \frac{\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 + \left[\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i - \mu\right]^2}{\epsilon^2}$$

$$= \frac{0 + (\mu - \mu)^2}{\epsilon^2} = 0 \qquad \text{for } \epsilon \neq 0$$

and hence

$$\lim_{n\to\infty} P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right| > \epsilon\right) = 0 \qquad \forall \epsilon > 0$$

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Question 3

3 We are given the random variables

$$U = X^2 + Y^2, \quad V = \frac{X}{\sqrt{X^2 + Y^2}}$$

Letting $u = x^2 + y^2$ and $v = \frac{x}{\sqrt{x^2+y^2}}$, we see that $y = \pm\sqrt{u - uv^2}$ which does not satisfied the condition that x and y must be uniquely solved in terms of u and v. Hence the Changes of Variables Method cannot apply here.

By letting random variable $Z = Y^2$, we have $u = x^2 + z$ and $v = \frac{x}{\sqrt{x^2+z}}$, in which we can get the unique solutions of $z = u - uv^2$ and $x = \sqrt{u}v$. Hence we need to find out the density function of r.v. Z and then apply the Changes of Variables Method to obtain r.v. U and V.

$$P(Z < z) = P(Y^2 < z) = P\left(-\sqrt{z} < Y < \sqrt{z}\right) = F_Y\left(\sqrt{z}\right) - F_Y\left(-\sqrt{z}\right)$$

$$\therefore \quad f_Z(z) = \frac{1}{2\sqrt{z}} f_Y\left(\sqrt{z}\right) + \frac{1}{2\sqrt{z}} f_Y\left(-\sqrt{z}\right) = \frac{1}{2\sqrt{2\pi z}\sigma} e^{-\frac{z}{2\sigma^2}} + \frac{1}{2\sqrt{2\pi z}\sigma} e^{-\frac{z}{2\sigma^2}} = \frac{1}{\sqrt{2\pi z}\sigma} e^{-\frac{z}{2\sigma^2}}$$

Suppose $g_u(x,z) = x^2 + z$ and $g_v(x,z) = \frac{x}{\sqrt{x^2+z}}$, we have

$$\frac{\partial g_u}{\partial x} = 2x \qquad \frac{\partial g_u}{\partial z} = 1$$

$$\frac{\partial g_v}{\partial x} = \frac{\sqrt{x^2+z} - (x)\left(\frac{1}{2}\right)\frac{1}{\sqrt{x^2+z}}(2x)}{x^2+z} = \frac{z}{(x^2+z)^{\frac{3}{2}}}$$

$$\frac{\partial g_v}{\partial z} = \frac{-(x)\left(\frac{1}{2}\right)\frac{1}{\sqrt{x^2+z}}}{x^2+z} = \frac{-x}{2(x^2+z)^{\frac{3}{2}}}$$

Hence, the Jacobian

$$J(x,z) = \frac{-x(2x)}{2(x^2+z)^{\frac{3}{2}}} - \frac{z}{(x^2+z)^{\frac{3}{2}}} = \frac{-(x^2+z)}{(x^2+z)^{\frac{3}{2}}} = \frac{-1}{\sqrt{x^2+z}} = -\frac{1}{\sqrt{u}}$$

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As the joint density function of X and Y is

$$f_{X,Z}(x,z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi z}\sigma} e^{-\frac{z}{2\sigma^2}} = \frac{1}{2\pi\sqrt{z}\sigma^2} e^{-\frac{x^2+z}{2\sigma^2}} = \frac{1}{2\pi\sqrt{u(1-v^2)}\sigma^2} e^{-\frac{u}{2\sigma^2}}$$

We see that the joint density function of U and V is given by

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Z}(x,z)|J(x,z)|^{-1} \\ &= \frac{1}{2\pi\sqrt{u(1-v^2)}\sigma^2} e^{-\frac{u}{2\sigma^2}} \cdot \sqrt{u} \\ &= \frac{1}{2\sigma^2} e^{-\frac{u}{2\sigma^2}} \cdot \frac{1}{\pi\sqrt{1-v^2}} \qquad 0 < u < \infty, \qquad -1 < v < 1 \end{aligned}$$

As this joint density factors into the marginal densities for U and V, we obtain that U and V are independent random variables, with U being an exponential distribution with parameter $\frac{1}{2\sigma^2}$, and V with density function of

$$f_v(v) = \frac{1}{\pi\sqrt{1-v^2}} \qquad -1 < v < 1$$

Question 4

4a Given that

$$Y_n = \max\{X_1, \dots, X_n\}$$

We can deduce the following probability:

$$\begin{aligned} P(Y_n < a) &= P([\max\{X_1, \dots, X_n\}] < a) \\ &= P(\text{ all of } \{X_1, \dots, X_n\} < a) \\ &= P\{(X_1 < a) \cap (X_2 < a) \cap \dots \cap (X_n < a)\} \\ &= \prod_{i=1}^{n} P(X_i < a) \\ &= \begin{cases} 0 & a \le 0 \\ \left(\frac{a}{\theta}\right)^n & 0 < a < \theta \\ 1 & a \ge \theta \end{cases} \end{aligned}$$

Suppose $k_n = \frac{n}{\theta}$, which satisfies $k_n \uparrow +\infty$ as $n \uparrow +\infty$,

$$\begin{aligned} \lim_{n\to\infty} P(k_n(\theta - Y_n) \le x) &= \lim_{n\to\infty} P\left(\frac{n}{\theta}(\theta - Y_n) \le x\right) \\ &= \lim_{n\to\infty} P\left(Y_n \ge \theta\left(1 - \frac{x}{n}\right)\right) \\ &= 1 - \lim_{n\to\infty} P\left(Y_n < \theta\left(1 - \frac{x}{n}\right)\right) \\ &= 1 - \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n \\ &= 1 - e^{-x} \qquad \text{for } 0 < x < \infty \end{aligned}$$

Hence,

$$G(x) = 1 - e^{-x} \qquad k_n = \frac{n}{\theta}$$

$G(x)$ is the continuous distribution function of an exponential random variable with parameter $\lambda = 1$.

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4b From Part 4(a), we know

$$P(Y_n < a) = \begin{cases} 0 & a \leq 0 \\ \left(\frac{a}{\theta}\right)^n & 0 < a < \theta \\ 1 & a \geq \theta \end{cases}$$

Hence,

$$P\left(\lim_{n\to\infty} Y_n < a\right) = \begin{cases} 0 & a \leq 0 \\ \lim_{n\to\infty} \left(\frac{a}{\theta}\right)^n & 0 < a < \theta \\ 1 & a \geq \theta \end{cases}$$
$$= \begin{cases} 0 & a < \theta \\ 1 & a \geq \theta \end{cases}$$

We know that for all $\epsilon > 0$,

$$\begin{aligned}
\lim_{n\to\infty} P(|Y_n - \theta| > \epsilon) &= P\left(\left|\lim_{n\to\infty} Y_n - \theta\right| > \epsilon\right) \\
&= P\left(\lim_{n\to\infty} Y_n - \theta > \epsilon\right) + P\left(\lim_{n\to\infty} Y_n - \theta < -\epsilon\right) \\
&= P\left(\lim_{n\to\infty} Y_n > \theta + \epsilon\right) + P\left(\lim_{n\to\infty} Y_n < \theta - \epsilon\right) \\
&= \left[1 - P\left(\lim_{n\to\infty} Y_n < \theta + \epsilon\right)\right] + \left[P\left(\lim_{n\to\infty} Y_n < \theta - \epsilon\right)\right]
\end{aligned}$$

We see that

$$\left[1 - P\left(\lim_{n\to\infty} Y_n < \theta + \epsilon\right)\right] = 1 - 1 = 0 \qquad \forall \epsilon > 0$$

and

$$\left[P\left(\lim_{n\to\infty} Y_n < \theta - \epsilon\right)\right] = 0 \qquad \forall (\theta - \epsilon) < \theta \to \epsilon > 0$$

Hence,

$$\lim_{n\to\infty} P(|Y_n - \theta| > \epsilon) = 0 \qquad \forall \epsilon > 0$$

Yes. It is proven to be true. (Q.E.D.)