ST2131 Probability Homework 1

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Solutions

Question 1

1a There are 3 possible outcomes if members A and B do not want to serve together: Both A and B are not chosen, A is chosen but B is not, or B is chosen but A is not.

Case 1:
Both A and B are not chosen. Hence we have 8 candidates left for the first position (president), and then 7 candidates left for the second position (secretary), and subsequently 6 candidates left for the last position (bursar).

$$\therefore \text{Number of Choices } = 8 \cdot 7 \cdot 6$$

Case 2:
A is chosen but B is not. If A is chosen, we left 2 more position to be chosen between the 8 remaining candidates. Since there are 3 possible positions A can be chosen as, the total number of choices is multiplied by 3.

$$\therefore \text{Number of Choices} = 3 \cdot 8 \cdot 7$$

Case 3:
B is chosen but A is not. The number of choices is similar to that of case 2.

$$\therefore \text{Number of Choices} = 3 \cdot 8 \cdot 7$$

Overall:
The total number of choices will be the summation of the number of choices of all 3 cases.

$$\therefore \text{Total Number of Choices } = 8 \cdot 7 \cdot 6 + 3 \cdot 8 \cdot 7 + 3 \cdot 8 \cdot 7 = 672$$

1b From part 1a, we know that the number of choices for member A to be chosen to hold any of these 3 positions is $3 \cdot 8 \cdot 7$ (refer to case 2).

$$\therefore \text{Chance of member } A = \frac{\text{Number of choices for member } A \text{ to be chosen}}{\text{Total number of choices}}$$
$$= \frac{3 \cdot 8 \cdot 7}{672} = \frac{1}{4}$$

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Question 2

2 Given that Z is a standard normal random variable with cumulative distribution function Φ. The cumulative distribution function of random variable $Y = -\ln(1-\Phi(\mathrm{Z}))$ is

$$\begin{aligned} F_Y(y) = P(Y < y) &= P\{-\ln(1-\Phi(\mathrm{Z})) < y\} \\ &= P\{\ln(1-\Phi(\mathrm{Z})) > -y\} \\ &= P\{1-\Phi(\mathrm{Z}) > e^{-y}\} \\ &= P\{\Phi(\mathrm{Z}) < 1-e^{-y}\} \\ &= P\{Z < \Phi^{-1}(1-e^{-y})\} \\ &= \Phi\{\Phi^{-1}(1-e^{-y})\} \\ &= 1-e^{-y} \end{aligned}$$

We know that $0 < \Phi(\mathrm{Z}) < 1$, it follows that $0 < y < \infty$ from the transformation. Hence, the distribution of the random variable Y is

$$f_y(y) = F_y'(y) = e^{-y}, \qquad 0 < y < \infty$$

Question 3

3 Let R_n be the event that a red ball is drawn from box B_n. We have

$$P(R_n) = p^{\frac{a^n}{n!}}$$

for $0 < p < 1, 0 < a < 1, n = 1,2, \dots$

The event of all drawn balls from each boxes are red is $R_1 \cap R_2 \cap \dots \cap R_n \cap \dots$

Note that these sets are mutually independent, we now have

$$\begin{aligned} P(R_1 \cap R_2 \cap \dots) &= P(R_1) \cdot P(R_2) \cdot P(R_3) \cdot \dots \\ &= p^{\frac{a^1}{1!}} \cdot p^{\frac{a^2}{2!}} \cdot p^{\frac{a^3}{3!}} \cdot \dots \\ &= p^{\frac{a^1}{1!}+\frac{a^2}{2!}+\frac{a^3}{3!}+\cdots} \\ &= p^{\left(1+\frac{a^1}{1!}+\frac{a^2}{2!}+\frac{a^3}{3!}+\cdots\right)-1} \\ &= p^{e^a-1} \end{aligned}$$

We have applied the definition of mutually independence event in probability in this solution. It says that if event $\{R_1, R_2, \dots, R_n\}$ are mutually independent, the probability of the intersection of any events is equal to the product of the probability of each individual event.

Finally, using the continuity properties of probabilities, we have that the desired probability

$$P\left(\bigcap_{i=1}^{\infty} R_i\right) = P\left(\lim_{n\to\infty} \bigcap_{i=1}^{n} R_i\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} R_i\right) = \lim_{n\to\infty} \prod_{i=1}^{n} P(R_i)$$

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Question 4

4a Let B_k be the event $\bigcup_{n=k}^{\infty} E_n$ for $k = 1,2, \dots$, we say that if x is belong to an infinite number of E_n , then

$$x \in B_k \ \ \forall k$$

We can show that this is indeed true by contradiction. Suppose that x does not belong to an event B_r where $B_r = \bigcup_{n=r}^{\infty} E_n$, then x can only exist in finitely many event E_n where $0 < n < r$. It contradicts the assumption that x is belong to an infinite number of E_n.

Since $x \in B_k$ for all k, it follows that

$$x \in \bigcap_{k=1}^{\infty} B_k$$

which is

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

Hence, we have

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = \{x \in \Omega : x \text{ belongs to an infinite number of } E_n\}$$

(Q.E.D.)

4b Let $A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$ and $B_k = \bigcup_{n=k}^{\infty} E_n$ for all k. We see that $A \subseteq B_k$ for all k, it follows that $P(A) \leq P(B_k)$ for all k.

By using Boole's inequality, we know that

$$P(B_k) = P\left(\bigcup_{n=k}^{\infty} E_n\right) \leq \sum_{n=k}^{\infty} P(E_n)$$

From $\sum_{n=1}^{\infty} P(E_n) < \infty$, by the theory of convergence of series, we know that the series $\{P(E_1), P(E_2), \dots\}$ converges. In the other words, the tail sum of the series converge to 0. Hence $P(B_k) \to 0$ as $k \to \infty$. Therefore, from the inequality $P(A) \leq P(B_k)$ for all k,

$$P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) = P(A) = 0$$

(Q.E.D.)