EE6105 - Nonlinear Dynamics and Control
Homework I-3

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1. (d) Suppose we have a SISO system,

$$\begin{aligned} \dot{x} &= Ax + Bu, && x \in \mathbb{R}^n, A \text{ Hurwitz} \\ y &= Cx, && y, u \in \mathbb{R} \\ u &= -\psi(t, y), && \psi \in [-k, 0], k > 0 \end{aligned}$$

Let $V(x) = x^T P x$. Then,

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= [x^T A^T - \psi B^T] P x + x^T P [Ax - B\psi] \\ &= x^T [A^T P + PA] x - 2x^T P B \psi \end{aligned}$$

Sector nonlinearity implies that $-2[ky + \psi]\psi \geq 0$,

$$\begin{aligned} \dot{V}(x) &\leq x^T [A^T P + PA] x - 2x^T P B \psi - 2[ky + \psi]\psi \\ &= x^T [A^T P + PA] x + 2x^T [-kC^T - PB]\psi - 2\psi^2 \end{aligned}$$

Now, find $P > 0$, L and $\varepsilon > 0$ such that

$$\begin{aligned} A^T P + PA &= -L^T L - \varepsilon P \\ PB &= -kC^T - L^T \sqrt{2}, \end{aligned}$$

then,

$$\begin{aligned} \dot{V}(x) &\leq -\varepsilon x^T P x - x^T L^T L x + 2x^T L^T \sqrt{2}\psi - 2\psi^2 \\ &= -\varepsilon x^T P x - [Lx - \sqrt{2}\psi]^T [Lx - \sqrt{2}\psi] \\ &\leq -\varepsilon x^T P x \\ &< 0 \end{aligned}$$

According to Kalman-Yakubovich-Popov (Positive Real) Lemma, we can construct a transfer matrix

$$Z(s) = -kC(sI - A)^{-1}B + I$$

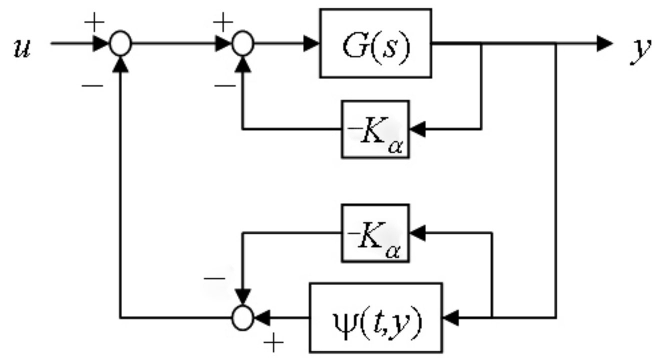
where $Z(s)$ is strictly positive real (SPR). We then have

$$\mathbf{Re}[-kG(j\omega) + I] > 0 \iff \mathbf{Re}[-G(j\omega)] > -\frac{1}{k}.$$

For the special case $-k = \alpha$, we have the system stable in $(\alpha, 0)$, if G is Hurwitz and

$$\mathbf{Re}[-G(j\omega)] > \frac{1}{\alpha}.$$

(e) We can apply similar argument as shown in the lecture notes: Consider the loop transformation



where K_α is chosen such that

$$G_T(s) = G(s)[I - K_\alpha G(s)]^{-1}$$

is asymptotically stable. Now let the new non-linearity to be $\psi_T(t,y) = \psi(t,y) + K_\alpha y$. Suppose ψ is a $[-K_\beta, -K_\alpha]$-sector non-linearity where $K_\beta > K_\alpha$, we have

$$\begin{aligned} \psi \in [-K_\beta, -K_\alpha] \quad &\Longrightarrow \quad [\psi + K_\beta y][\psi + K_\alpha y] \le 0 \\ &\Longrightarrow \quad \psi_T[\psi_T - K_\alpha y + K_\beta y] \le 0 \\ &\Longrightarrow \quad \psi_T[\psi_T + (K_\beta - K_\alpha)y] \le 0 \\ &\Longrightarrow \quad \psi_T \in [-K, 0] \qquad \text{for } K = K_\beta - K_\alpha > 0 \end{aligned}$$

Thus, applying the conclusion from (d) to the system $G_T(s)$ and the sector non-linearity $\psi_T \in [-K, 0]$, the sufficient condition for absolutely stable of the closed-loop is $G_T(s)$ Hurwitz and $Z_T(s) = I - KG_T(s)$ strictly positive real.

Here, since K_α was taken to be the boundary of the sector, we need to verity the stability of $G_T(s)$. Let $K_\alpha = -\beta$, and $K_\beta = -\alpha$, we have

$$G_T(s) = G(s)[1 + \beta G(s)]^{-1}$$

and since

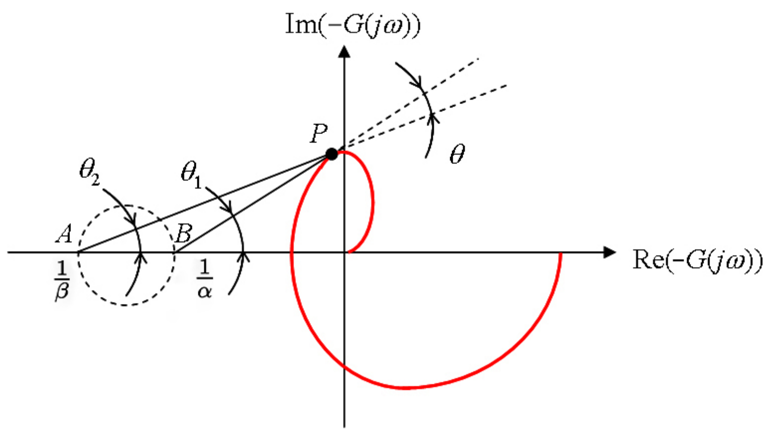
$$\begin{aligned} Z_T(s) &= 1 - (\beta - \alpha)\frac{G(s)}{1 + \beta G(s)} \\ &= \frac{1 + \alpha G(s)}{1 + \beta G(s)} \end{aligned}$$

is strictly positive real. we have

$$\mathbf{Re}\Big[\frac{1 + \alpha G(j\omega)}{1 + \beta G(j\omega)}\Big] > 0, \quad \forall \omega > 0.$$

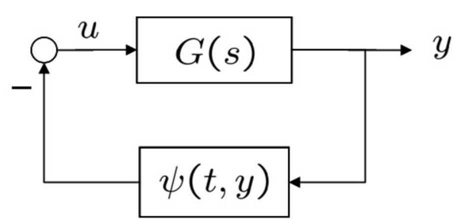
For $\alpha < \beta < 0$, base on the diagram shows below, we have

$$\begin{aligned} \mathbf{Re}\Big[\frac{1 + \alpha G(j\omega)}{1 + \beta G(j\omega)}\Big] > 0 \quad &\Longleftrightarrow \quad \mathbf{Re}\Big[\frac{-1/\alpha - G(j\omega)}{-1/\beta - G(j\omega)}\Big] > 0 \\ &\Longleftrightarrow \quad \mathbf{Re}\Big[\frac{BP}{AP}\Big] > 0 \\ &\Longleftrightarrow \quad \cos(\theta_1 - \theta_2) > 0 \\ &\Longleftrightarrow \quad \theta \notin \Big[\frac{\pi}{2}, \frac{3\pi}{2}\Big] \\ &\Longleftrightarrow \quad P \text{ not in the circle } D(-\beta, -\alpha) \end{aligned}$$



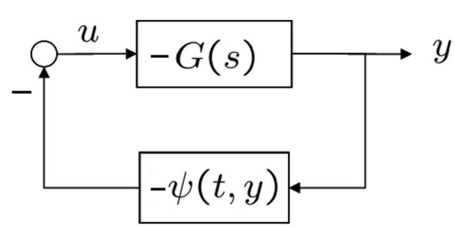
Since $G_T(s) = G(s)[1 - \beta(-G(s))]^{-1}$ has to be Hurwitz for the result to hold, it implies that $-G(j\omega)$ needs to encircle $(1/\beta, 0j)$ or disk $D(-\beta, -\alpha)$ (since $-G(j\omega)$ does not enter the disk) counter-clockwise a number of times equal to the number of unstable poles (ORHP) of $G(s)$.

Alternative Method: Simple and Intuitive Approach



Consider the system as shown in the diagram above, we are given that the (a), (b), and (c) conditions in Exercise 1 Question 1 hold true as a result of Circle Criterion for sector non-linearity of $\psi \in (\alpha, \beta)$.

Now let us consider an equivalent system, where $G_T(s) = -G(s)$ and $\psi_T(t, y) = -\psi(t, y)$, as shown in figure below:



We can see that the closed-loop system remains unchange, with the new sector non-linearity of $\psi_T \in (-\beta, -\alpha)$. Now by assigning

$$\begin{aligned} -\alpha &= 0 \\ -\beta &= \alpha_T \end{aligned}$$

then by the second Circle Criterion, we can conclude that the closed-loop system is absolutely stable in $(\alpha_T, 0)$ if $G(s)$ is Hurwitz and $G_T(j\omega) = -G(j\omega)$ remains on the right of

$$\mathbf{Re}[z] = -\frac{1}{\beta} = \frac{1}{\alpha_T}.$$

Similarly, by assigning

$$\begin{aligned} -\alpha &= \beta_T \\ -\beta &= \alpha_T \end{aligned}$$

then by the first Circle Criterion, we can conclude that the closed-loop system is absolutely stable in (α_T, β_T), $\alpha_T < \beta_T < 0$, if $G_T(j\omega) = -G(j\omega)$ does not enter $D(-\beta_T, -\alpha_T)$ and encircles it counter-clockwise a number of times equal to the number of ORHP poles of $G(s)$.

2. (a) At equilibrium, $\dot{x}_1 = \dot{x}_2 = 0$, we have

$$\begin{aligned} x_1 &= h(x_1 + x_2) \\ x_2 &= x_1 - 2h(x_1 + x_2) = -h(x_1 + x_2) \end{aligned}$$

Thus we have $x_1 = -x_2$ at equilibrium. Since $y = x_1 + x_2$, we have $y = 0$. We are given $h(0) = 0$, and thus

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

The origin is the unique equilibrium point.

(b) We rewrite the system in the following form,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x \\ u &= -h(y) \end{aligned}$$

Then,

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{3s+4}{(s+1)^2} \end{aligned}$$

and $h(y)$ is a sector non-linearity in the range of

$$0 \le yh(y) \le cy, \quad \forall y$$

Divide by y^2,

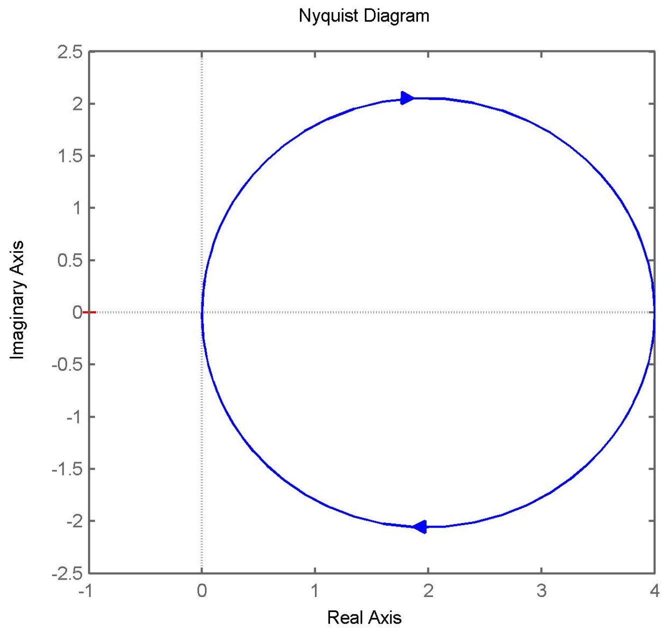
$$0 \le \frac{h(y)}{y} \le \frac{c}{y}, \quad \forall y$$

We know $h(y) = 0$ for all $|y| \le a_1$, and $h(y) \le c$ for all $|y| > a_1$, we can form a tight bound

$$0 \le \frac{h(y)}{y} \le \frac{c}{y} \le \frac{c}{a_1}, \quad \forall y$$

and thus we finally have sector non-linearity in $h \in [0, \frac{c}{a_1}]$. Since $G(s)$ is Hurwitz here, by applying Circle Criterion, the closed-loop system will be globally asymptotically stable if $G(j\omega)$ remains on the right of $\mathbf{Re}[z] = -\frac{1}{c/a_1}$.

For the case of $c = 0$, there is no feedback and therefore the system is stable. Otherwise, since both $c, a_1 > 0$, we also have the Nyquist plot of $G(j\omega)$ stays at the closed right-half-plane (see figure below), and therefore strictly on the right of $-\frac{1}{c/a_1}$.



3. It is given that

$$\psi^T(y)[\psi(y) - Ky] \leq 0,$$

expand it, we have

$$\begin{aligned} \psi^T(y)Ky &\geq \psi^T(y)\psi(y) \geq 0 \\ (K\psi(y))^T y &\geq 0 \\ \left(\frac{\partial F}{\partial y}\right)^T y &\geq 0 \\ \frac{\partial F}{\partial y} &\geq 0, \quad \forall y \geq 0. \end{aligned}$$

Since the gradient of $F(y)$ is always greater or equal to 0 in this range, we know that $F(y)$ is a monotonically increasing function for $y \geq 0$. We thus conclude that $F(y) \geq F(0)$ for all $y > 0$.

Now consider a Lyapunov function candidate with $P > 0$ and $\eta > 0$,

$$\begin{aligned} V(x) &= x^t Px + 2\eta \int_0^y \psi^T(v)K dv \\ &= x^T Px + 2\eta \int_0^y (K\psi(v))^T dv \\ &= x^T Px + 2\eta \int_0^y \left(\frac{\partial F(v)}{\partial v}\right)^T dv \\ &= x^T Px + 2\eta(F(y) - F(0)) \end{aligned}$$

Since we know $F(y) \geq F(0)$ earlier, we then have

$$V(x) \geq x^T Px > 0$$

The Lyapunov function candidate here is exactly the same as the given one in the derivation of multivariable Popov criterion in the lecture note. Thus we have the rest of the proof shown in the lecture note.

4. Consider the storage function $S = V + \frac{1}{2}y^T y$. We have $S(0) = V(0) = 0$ and $S \geq V \geq 0$, we thus only need to show that $\dot{S} \leq v^T y$ for passivity.

$$\begin{aligned}\dot{S} &= \dot{V} + y^T \dot{y} \\ &= \dot{V} + y^T \dot{x}_2 \\ &= \dot{V} + y^T[-(L_g V)^T + v] \\ &\leq y^T[-(L_g V)^T + v] \qquad \text{since } \dot{V} \leq 0\end{aligned}$$

By assuming that the feedback transformation was a negative feedback, we have $L_g V > 0$. We thus have

$$\dot{S} \leq y^T v = v^T y$$

Hence the feedback system is passive.

Now let Z be the largest invariant set of the system contained in $\{x|y = 0\}$. Obviously, we have $Z = \{x|x_2 = 0\}$. At this condition, the system reduces to its zero-dynamics system Σ_{zd}. Since we are given that the zero-dynamics are asymptotically stable, H is zero-state detectable (ZSD). Then by using the passivity and stability theorem, since H is passive, S is C^1, and y is C^1, we have the feedback $u = -y = -x_2$ achieves asymptotic stability of the closed-loop system.

5. Suppose there is a periodic solution $y(t) = a \sin \omega t$, taking the first-order harmonic terms, we have the describing function

$$N(a) = \frac{1}{\pi a}\int_{\pi}^{\pi} \psi(a \sin \omega t) \sin \omega t dt + j\frac{1}{\pi a}\int_{\pi}^{\pi} \psi(a \sin \omega t) \cos \omega t dt.$$

Since $\psi(y) = y^5$ is an odd function, we have

$$\begin{aligned}N(a) &= \frac{1}{\pi a}\int_{\pi}^{\pi} \psi(a \sin \omega t) \sin \omega t dt \\ &= \frac{4}{\pi a}\int_{0}^{\pi/2} a^5 \sin^6 \omega t dt \\ &= \frac{4a^4}{\pi}\int_{0}^{\pi/2} \sin^6 \omega t dt \\ &= \frac{4a^4}{\pi} \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \\ &= \frac{5a^4}{8}\end{aligned}$$

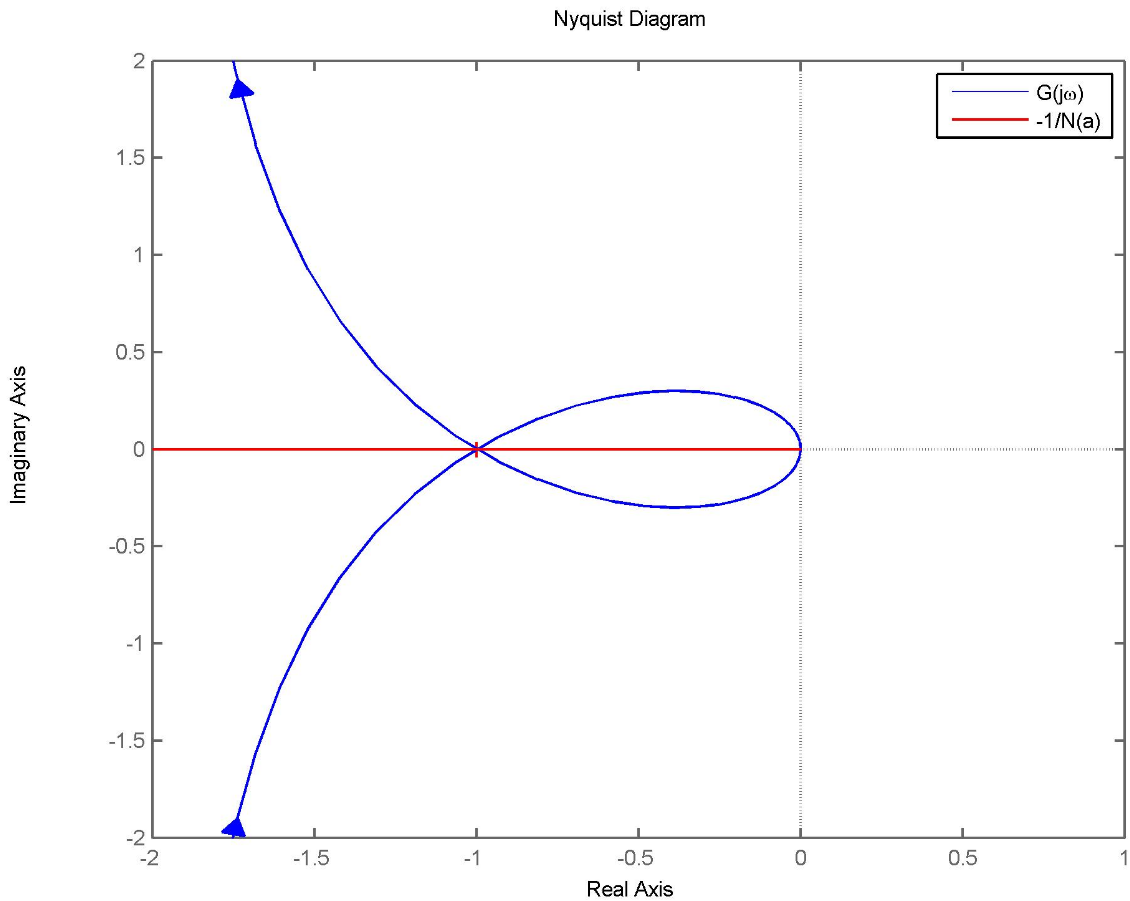
To check the possibility of periodic solution, we solve the first-order harmonic balance

$$\begin{aligned}1 + N(a)G(j\omega) &= 0 \\ G(j\omega) &= -\frac{1}{N(a)} \\ \frac{1 - j\omega}{-\omega^2 + j\omega} &= -\frac{8}{5a^4}\end{aligned}$$

Solving them, we have

$$\begin{aligned}a &= \sqrt[4]{\frac{8}{5}} \\ \omega &= 1\end{aligned}$$

Thus, there is a possible periodic solution, with the fundamental frequency of 1 rad/s and amplitude of $\sqrt[4]{\frac{8}{5}}$. To check the stability of the periodic solution, we see the following Nyquist plot of $G(j\omega)$:



We see that for any loop gains, periodic solutions will occur as $G(j\omega)$ will intersect $-\frac{1}{N(a)}$ definitely. Therefore the periodic solutions are stable.