# EE6105 - Nonlinear Dynamics and Control
# Homework I-2

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1. Suppose $L$ is a Lipschitz constant for $f$. For any real number $\varepsilon > 0$, there exists a positive real number $\delta$ where

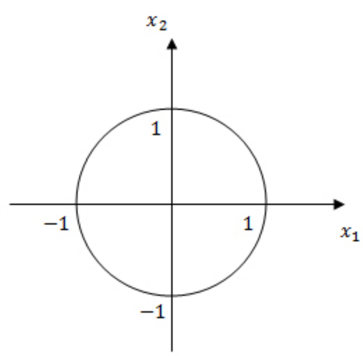
$$\delta = \frac{\varepsilon}{L}.$$

Since $f$ is Lipchitz on $W$, for any $x, y \in W$ with $||x - y|| < \delta$, we have

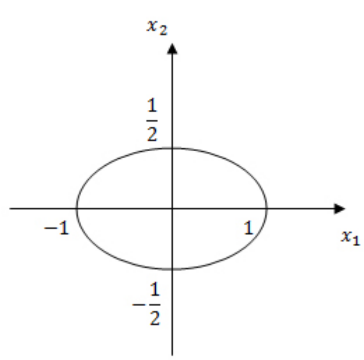
$$||f(x) - f(y)|| \leq L||x - y|| < L\delta = \varepsilon$$

It shows that $f$ is uniformly continuous on $W$.

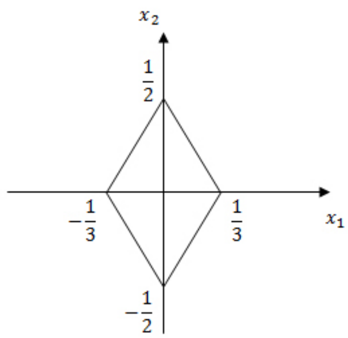
2. (a)



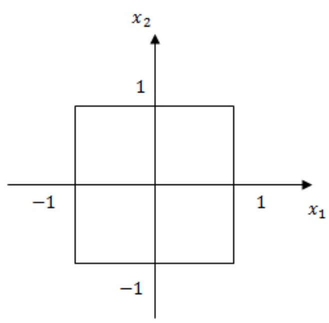

(b)



(c)




(d)




3. A linear algebraic equation $Ax = b$ has a unique solution if and only if $A$ is non-singular. To show that $A$ is non-singular, let $\lambda$ be an eigenvalue of $A$ with corresponding eigenvector,

$$v = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^T.$$

Let $v_i$ be the largest value in terms of magnitude in the vector $v$, we have $|v_i| > 0$. Since $v$ is an eigenvector corresponds to eigenvalue $\lambda$, we know

$$Av = \lambda v. \tag{1}$$

Taking the $i$-th row of Equation (1),

$$\sum_j a_{ij} v_j = \lambda v_i.$$

Splitting the sum, we get

$$\sum_{j \neq i} a_{ij} v_j = \lambda v_i - a_{ii} v_i.$$

Dividing both sides by $v_i$ and take the absolute value,

$$|\lambda - a_{ii}| = \left| \frac{\sum_{j \neq i} a_{ij} v_j}{v_i} \right| \leq \sum_{j \neq i} |a_{ij}|$$

since $|v_j/v_i| \leq 1$ for $j \neq i$. This is also called Gershgorin's Circle Theorem.

Now suppose $A$ is a singular matrix, there exists at least one zero eigenvalue $\lambda = 0$. By applying Gershgorin's Circle Theorem, we have

$$|\lambda - a_{ii}| = |a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$$

which contradict the inequality $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$. Therefore matrix $A$ is non-singular.

4. We can prove this theorem by contradiction. First we need to solve the differential inequality $\dot{V}(x) \geq \lambda V(x)$. Let us define a function $Z(x)$ by

$$Z(x) = \dot{V}(x) - \lambda V(x)$$

The differential inequality implies that $Z(x)$ is semi-positive. The solution of the first-order equation above is

$$V(x) = e^{\lambda t} V(x_0) + \int_0^t e^{\lambda(t-\tau)} Z(x) d\tau$$

Because the second term in the right-hand-side of the above equation is semi-positive, we have

$$V(x) \geq e^{\lambda t} V(x_0), \quad \forall \lambda > 0. \tag{2}$$

Now let $R > 0$ so that $\beta_R \subset \Omega$, and let any $r > 0$ such that $r < R$. Then by the assumption of the system is stable, there exists $x_0 \in \beta_r$ such that $V(x_0) = a > 0$, and that the trajectory of $x$ will not exceed $\beta_R$.

Since $V$ is continuous and $\tilde{\beta}_R$ is compact, by applying Weierstrass Theorem, there exists $m \geq a > 0$ such that for all $x \in \tilde{\beta}_R$, $V(x) \leq m$.

Now, substituting $V(x_0) = a$ into Equation (2), we have

$$V(x) \geq e^{\lambda t} a$$

Let $e^{\lambda t} a = m$, we have $t = \frac{1}{\lambda} \ln \frac{m}{a}$. Thus, at time $t > \frac{1}{\lambda} \ln \frac{m}{a}$, we see that $V(x)$ must exceed $m$, which contradicts the assumption that $x(t)$ remains in $\tilde{\beta}_R$. Hence $x(t)$ must leave $\tilde{\beta}_R$.

5. (a) Differentiate $V$ w.r.t. $t$, we get

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1^2(k^2 - x_1^2 - x_2^2) + x_1x_2(k^2 + x_1^2 + x_2^2) - x_1x_2(k^2 + x_1^2 + x_2^2) + x_2^2(k^2 - x_1^2 - x_2^2) \\
&= (x_1^2 + x_2^2)(k^2 - x_1^2 - x_2^2)
\end{aligned}$$

For $k = 0$, the above equation simplifies to

$$\dot{V} = -(x_1^2 + x_2^2)^2 < 0$$

Since $V$ is positive definite and radially unbounded, $\dot{V}$ is negative definite, it verify that the system is globally asymptotically stable.

(b) For $k \neq 0$, $\dot{V} = 2Vk^2 - 4V^2$. We can see that for $V < \frac{k^2}{2}$, $\dot{V}$ will be positive. Let $\Omega$ be a compact region bounded by open ball $\beta_k$, and we know that $\dot{V}$ is positive definite in this region. Let $R > 0$ so that $\beta_R \in \Omega$, and $r > 0$ such that $r < R$, given any initial point $x_0$ in ball $\beta_r$, we see that the trajectory will exceed ball $\beta_R$ in finite time. Therefore the origin is unstable.

(c) Since $2V = x_1^2 + x_2^2$, we can express $\dot{V}$ as

$$\dot{V} \begin{cases} < 0 & \text{if } x_1^2 + x_2^2 > k^2 \\ = 0 & \text{if } x_1^2 + x_2^2 = k^2 \\ > 0 & \text{if } x_1^2 + x_2^2 < k^2 \end{cases}$$

We can see that any trajectory inside the ball $\beta_k$ will diverge from the origin to sphere $S_k$, whereas any trajectory outside the ball $\beta_k$ will converge to it. Since $V$ is not moving at sphere $S_k$, trajectory in sphere $S_k$ will stay in $S_k$ forever. Therefore we can conclude that $S_k$ is a limit cycle for this system.

6. (a) To prove positive-definiteness of a function, first,

$$\begin{aligned} V(t,0) &= \frac{1}{2}(a\sin 0 + 0)^2 + [1 + ag(t) - a^2](1 - \cos 0) \\ &= 0 \end{aligned}$$

Then define a function,

$$V_0(x) = \frac{1}{2}(a\sin x_1 + x_2)^2 + [1 + a\alpha - a^2](1 - \cos x_1)$$

Since $[1 + ag(t) - a^2] \geq [1 + a\alpha - a^2] > 1$ for all $t \geq 0$, we have shown that $V_0(x)$ is a positive-definite function such that $V(t,x) \geq V_0(x)$ for all $t \geq 0$. Thus $V(t,x)$ is a positive-definite function.

To prove $V(t,x)$ a decrescent function, the first condition is met as shown above. Now we define another function,

$$V_1(x) = \frac{1}{2}(a\sin x_1 + x_2)^2 + [1 + a\beta - a^2](1 - \cos x_1)$$

Since $[1+a\beta-a^2] \geq [1+ag(t)-a^2] > 1$ for all $t \geq 0$, we have shown that $V_1(x)$ is a positive-definite function such that $V(t,x) \leq V_1(x)$ for all $t \geq 0$. Thus $V(t,x)$ is a decrescent function.

(b) *to be done*

7. Since $g(t) \leq 1$ for all $t \geq 0$, we have

$$\begin{aligned}
\dot{V}(x,t) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= -2x_1^2 - x_2^2 + 2x_1x_2g(t) \\
&\leq -2x_1^2 - x_2^2 + 2x_1x_2 \\
&= -x_1^2 - (x_1 - x_2)^2 \\
&< 0
\end{aligned}$$

Thus $\dot{V}$ is negative-definite. Since $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is positive-definite and is radially unbounded, we can conclude that the system is globally asymptotically stable.