# EE5106R - Advanced Robotics
# Assignment 1

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1. (a) The endpoint's position can be easily determine from the figure as

$$\mathbf{x} = \begin{bmatrix} {}^0x_3 \\ {}^0y_3 \\ {}^0z_3 \end{bmatrix} = \begin{bmatrix} d_3\cos\theta_1 \\ d_3\sin\theta_1 \\ l_1 + d_2 \end{bmatrix}$$

Taking derivatives of the position vector, we have

$$\begin{aligned} d\mathbf{x} = \begin{bmatrix} dx_3 \\ dy_3 \\ dz_3 \end{bmatrix} &= \begin{bmatrix} -d_3\sin\theta_1 d\theta_1 + 0dd_2 + \cos\theta_1 dd_3 \\ d_3\cos\theta_1 d\theta_1 + 0dd_2 + \sin\theta_1 dd_3 \\ 0d\theta_1 + 1dd_2 + 0dd_3 \end{bmatrix} \\ &= \begin{bmatrix} -d_3\sin\theta_1 & 0 & \cos\theta_1 \\ d_3\cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} d\theta_1 \\ dd_2 \\ dd_3 \end{bmatrix} \end{aligned}$$

Jacobian matrix,

$$J = \begin{bmatrix} -d_3\sin\theta_1 & 0 & \cos\theta_1 \\ d_3\cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) $J$ is singular if its determinant is zero,

$$\begin{aligned} \det J &= -d_3\sin^2\theta_1 - d_3\cos^2\theta_1 \\ &= -d_3 \equiv 0 \end{aligned}$$

Therefore, $d_3 = 0$ is the singular configuration for this manipulator. Here, however, an assumption is made such that the end effector of the robot manipulator is able to retreat to the point at $d_3 = 0$.

(c) At $\theta = (0, 1, 1)^T$, Jacobian matrix can be expressed as

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The equivalent joints' torques or forces corresponding to the endpoint force, $F = (1, 2, 3)^T$, is

$$\begin{bmatrix} \tau_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

(d) At $\theta = (0, 1, 0)^T$, Jacobian matrix can be expressed as

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We have

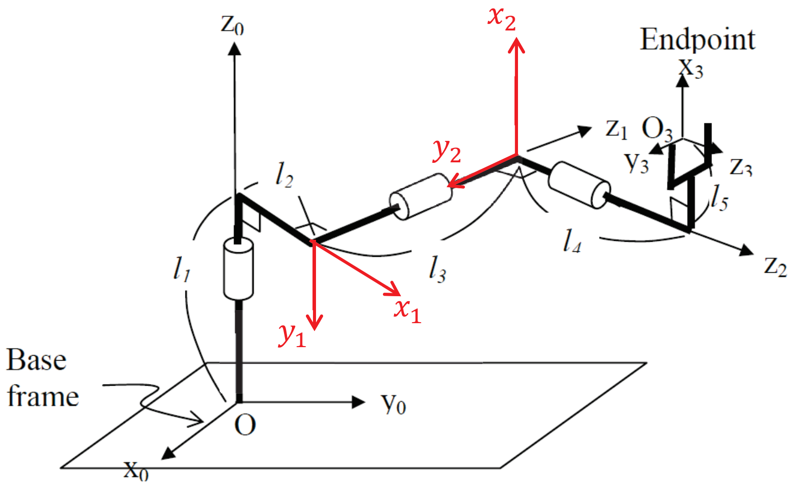
$$\begin{bmatrix} \tau_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 0 \\ F_z \\ F_x \end{bmatrix}$$

From the derivation above, we see that the equivalent joints' torques or forces is independent of the force at $y$ component, i.e. $F_y$. Therefore, when applying a force vector at

$$^0\hat{F} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

it requires zero equivalent joints' torques or forces.

2. (a) Link coordinate systems is assigned as follows:




| Link number | $\theta_i$ | $d_i$ | $a_i$ | $\alpha_i$ |
|---|---|---|---|---|
| 1 | $\theta_1$ | $l_1$ | $l_2$ | $-\frac{\pi}{2}$ |
| 2 | $\theta_2$ | $l_3$ | 0 | $-\frac{\pi}{2}$ |
| 3 | $\theta_3$ | $l_4$ | $l_5$ | 0 |

where $\theta_1, \theta_2, \theta_3$ are joint variables.

(b) From the equation (2-1) in Chapter 1 slides, we have

$$^{i-1}_{\ \ i}A = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Substituting the value for each variables, we have

$$
\begin{aligned}
{}^0_1A &= \begin{bmatrix} c_1 & 0 & -s_1 & l_2c_1 \\ s_1 & 0 & c_1 & l_2s_1 \\ 0 & -1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^1_2A &= \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
{}^2_3A &= \begin{bmatrix} c_3 & -s_3 & 0 & l_5c_3 \\ s_3 & c_3 & 0 & l_5s_3 \\ 0 & 0 & 1 & l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}
$$

where $c_*, s_*$ correspond to $\cos(\theta_*)$ and $\sin(\theta_*)$ respectively.

(c) $b_0$ is a unit vector pointing at $z_0$ direction, $b_1$ is the third column of ${}^0_1A$, while $b_2$ is the third column of ${}^0_1A^1_2A$.

$$
\begin{aligned}
b_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\
b_1 &= \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}, \\
b_2 &= \begin{bmatrix} -c_1s_2 \\ -s_1s_2 \\ -c_2 \end{bmatrix}.
\end{aligned}
$$

(d) For $l_1 = l_2 = l_4 = 0$ and $l_3 = l_5 = 1$, we have

$$
\begin{aligned}
{}^0_1A^1_2A^2_3A &= \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & c_3 \\ s_3 & c_3 & 0 & s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -s_1 \\ s_1c_2 & -c_1 & -s_1s_2 & c_1 \\ -s_2 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & c_3 \\ s_3 & c_3 & 0 & s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

End point coordinate express in frame 0 will be the last column of the above derivation, which is

$$
{}^0e = \begin{bmatrix} c_1c_2c_3 + s_1s_3 - s_1 \\ s_1c_2c_3 - c_1s_3 + c_1 \\ -s_2c_3 \end{bmatrix}
$$

3. Usually, the relationship between joint rates and end effector velocities of a 6 DOF manipulator is represented as

$$
d\mathbf{x}_E = J_E d\theta
$$

In this case, the computation of the singularities using this Jacobian matrix of 6 by 6 dimension is rather tedious. One approach is to split the problem of singularities to two different parts: wrist lock and arm lock.

As an example for the manipulator shown in the question, the position of the end effector to the position of the wrist (point $W$) is relatively fixed given the variables' value. Thus, the corresponding homogeneous and velocity transformation matrix from point $W$ to the end effector are known. Therefore the motion planning requirements specified at the end effector can be transformed to the wrist at point $W$ and vice versa.

Now with this relaxation, we can place the origin of the end effector at point $W$, results in $J_{12}$ being a zero matrix, where

$$\begin{aligned} d\mathbf{x}_W &= J_W d\theta \\ J_W &= \begin{bmatrix} J_{11} & 0_{3\times 3} \\ J_{21} & J_{22} \end{bmatrix} \end{aligned}$$

Now we can easily compute the singularities of $J_W$ as

$$\det J_W = \det J_{11} \det J_{22}$$

which is the combination of arm position singularity and wrist orientation singularity.