EE5103 - Computer Control Systems
Homework #2

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1. (a) Sample the system with sampling period h, we have

$$\begin{aligned}
\Phi &= e^{Ah} \\
&= L^{-1}(sI-A)^{-}1 \\
&= L^{-1}\begin{bmatrix} \frac{1}{s+10} & 0 \\ \frac{1}{s(s+10)} & \frac{1}{s} \end{bmatrix} \\
&= L^{-1}\begin{bmatrix} \frac{1}{s+10} & 0 \\ \frac{0.1}{s} - \frac{0.1}{s+10} & \frac{1}{s} \end{bmatrix} \\
&= \begin{bmatrix} e^{-10h} & 0 \\ 0.1(1-e^{-10h}) & 1 \end{bmatrix} \\
\Gamma &= \int_0^h e^{As} ds B \\
&= \int_0^h \begin{bmatrix} e^{-10s} & 0 \\ 0.1(1-e^{-10s}) & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} ds \\
&= \int_0^h \begin{bmatrix} 10e^{-10s} \\ 1-e^{-10s} \end{bmatrix} ds \\
&= \begin{bmatrix} 1-e^{-10h} \\ h-0.1(1-e^{-10h}) \end{bmatrix}
\end{aligned}$$

To simplify the following calculations, let us assign $e^{-10h} = \alpha$. Controllability matrix will then be

$$W_c = \begin{bmatrix} 1-\alpha & (1-\alpha)\alpha \\ h-0.1(1-\alpha) & 0.1(1-\alpha)^2 + h - 0.1(1-\alpha) \end{bmatrix}$$

The determinant of the controllability matrix is

$$\begin{aligned}
|W_c| &= 0.1(1-\alpha)^3 + h(1-\alpha) - 0.1(1-\alpha)^2 - h\alpha(1-\alpha) + 0.1\alpha(1-\alpha)^2 \\
&= h(1-\alpha)^2
\end{aligned}$$

The inverse will then be

$$W_c^{-1} = \frac{1}{h(1-\alpha)^2}\begin{bmatrix} 0.1(1-\alpha)^2 + h - 0.1(1-\alpha) & -(1-\alpha)\alpha \\ 0.1(1-\alpha) - h & 1-\alpha \end{bmatrix}$$

To design a deadbeat controller with $n = 2$, we have $A_m(z) = z^2$, and thus

$$\begin{aligned}
A_m(\Phi) &= \Phi^2 \\
&= \begin{bmatrix} \alpha & 0 \\ 0.1(1-\alpha) & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0.1(1-\alpha) & 1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha^2 & 0 \\ 0.1(1-\alpha)(1+\alpha) & 1 \end{bmatrix}
\end{aligned}$$

Finally, by using Ackermann's formula, we have the controller gain given by

$$\begin{aligned} L &= \frac{1}{h(1-\alpha)^2}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0.1(1-\alpha)^2+h-0.1(1-\alpha) & -(1-\alpha)\alpha \\ 0.1(1-\alpha)-h & 1-\alpha\end{bmatrix}\begin{bmatrix}\alpha^2 & 0 \\ 0.1(1-\alpha)(1+\alpha) & 1\end{bmatrix} \\ &= \frac{1}{h(1-\alpha)^2}\begin{bmatrix}0.1(1-\alpha)-h\alpha^2 & 1-\alpha\end{bmatrix} \\ &= \frac{1}{h(1-e^{-10h})^2}\begin{bmatrix}0.1(1-e^{-10h})-he^{-20h} & 1-e^{-10h}\end{bmatrix} \end{aligned}$$

and the deadbeat controller is given by

$$u(k) = -Lx(k)$$

(b) By assuming the maximum value of $u(k)$ is at $k=0$, we have

$$\begin{aligned} u(0) &= -\frac{1}{h(1-e^{-10h})^2}\begin{bmatrix}0.1(1-e^{-10h})-he^{-20h} & 1-e^{-10h}\end{bmatrix}\begin{bmatrix}x_1(0) \\ x_2(0)\end{bmatrix} \\ &= -\frac{1}{h(1-e^{-10h})^2}\begin{bmatrix}0.1(1-e^{-10h})-he^{-20h} & 1-e^{-10h}\end{bmatrix}\begin{bmatrix}1 \\ 0.5\end{bmatrix} \\ &= -\frac{0.6(1-e^{-10h})-he^{-20h}}{h(1-e^{-10h})^2} \end{aligned}$$

It is not hard to see that when h is getting larger, the magnitude of $u(0)$ is getting smaller. By simply letting $h=1$, we have

$$|u(0)| = |-0.6| < 1$$

2. (a) The augmented system becomes

$$\begin{aligned} \begin{bmatrix}x(k+1) \\ v(k+1)\end{bmatrix} &= \begin{bmatrix}0.5 & 1 & 1 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x(k) \\ v(k)\end{bmatrix} + \begin{bmatrix}1 \\ 1 \\ 0\end{bmatrix}u(k) \\ y(k) &= \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x(k) \\ v(k)\end{bmatrix} \end{aligned}$$

Now, be designing the controller $u(k) = -Lx(k) - L_v v(k)$, we have

$$\begin{bmatrix}x(k+1) \\ v(k+1)\end{bmatrix} = \begin{bmatrix}\Phi-\Gamma L & \Phi_{xv}-\Gamma L_v \\ 0 & 1\end{bmatrix}\begin{bmatrix}x(k) \\ v(k)\end{bmatrix}$$

Now we can choose L such that $\Phi - \Gamma L$ has desired poles. Supposed we just want a stable system, letting $L = \begin{bmatrix}1 & 1\end{bmatrix}$, the closed-loop system,

$$\begin{aligned} \Phi_{cl} &= \begin{bmatrix}0.5 & 1 \\ 0.2 & 0.8\end{bmatrix} - \begin{bmatrix}1 \\ 1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \\ &= \begin{bmatrix}-0.5 & 0 \\ -0.8 & -0.2\end{bmatrix} \end{aligned}$$

We can easily check that Φ_{cl} has poles at $z=-0.5$ and $z=-0.2$, which is stable. Now we check if it is possible to totally cancel the effects of disturbance, i.e. $\Phi_{xv} - \Gamma L_v = 0$. It is, however, not possible in this case. Fortunately, we can design L_v such that the transfer function from v to y is zero at steady state. Let $L_v = 1$, we can check the transfer function,

$$\begin{aligned} H_v(z) &= c(zI-(\Phi-\Gamma L))^-1(\Phi_{xv}-\Gamma L_v) \\ &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z+0.5 & 0 \\ 0.8 & z+0.2\end{bmatrix}^{-1}\begin{bmatrix}0 \\ -1\end{bmatrix} \\ &= 0 \end{aligned}$$

Thus, it shown that the influence of v can be eliminated.

(b) In this part, the disturbance v is not measurable. The state x is, however, measurable. Here we first design the controller to be identical to the previous part,

$$u(k) = -Lx(k) - L_v v(k)$$

where

$$\begin{aligned} L &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\ L_v &= 1 \end{aligned}$$

Now, we will need to design a reduced-order observer to observe the disturbance v, then the controller

$$u(k) = -Lx(k) - L_v \hat{v}(k) \tag{1}$$

will still work well if we can guarantee $\hat{v}(k) \to v(k)$ when k is large enough. Let us choose the observer to be

$$\hat{v}(k) = Kx(k) + z(k)$$

where $z(k)$ is the observer dynamic given by

$$z(k+1) = \bar{F}z(k) + \bar{G}x(k) + \bar{H}u(k) \tag{2}$$

The error dynamic

$$\begin{aligned} e(k) &= v(k) - \hat{v}(k) \\ e(k+1) &= v(k+1) - \hat{v}(k+1) \\ &= v(k) - Kx(k+1) - z(k+1) \\ &= v(k) - K\Phi x(k) - K\Phi_{xv} v(k) - K\Gamma u(k) - z(k+1) \end{aligned} \tag{3}$$

From (1) we have

$$\begin{aligned} z(k) &= \hat{v}(k) - Kx(k) \\ &= v(k) - e(k) - Kx(k) \end{aligned} \tag{4}$$

Substitute (3) and (4) into (2),

$$v(k) - Kx(k+1) - e(k+1) = \bar{F}(v(k) - e(k) - Kx(k)) + \bar{G}x(k) + \bar{H}u(k)$$

Arranging them properly, we have

$$\begin{aligned} e(k+1) = &\ \bar{F}e(k) + [\bar{F}L - \bar{G} - L\Phi]x(k) \\ &+ [1 - \bar{F} - L\Phi_{xv}]v(k) - [\bar{H} + L\Gamma]u(k) \end{aligned}$$

Thus, one can easily see that, by assigning

$$\begin{aligned} \bar{F}L - \bar{G} - L\Phi &= 0 \\ 1 - \bar{F} - L\Phi_{xv} &= 0 \\ \bar{H} + L\Gamma &= 0 \end{aligned}$$

or after rearranging them,

$$\begin{aligned} \bar{F} &= 1 - L\Phi_{xv} \\ \bar{G} &= \bar{F}L - L\Phi \\ \bar{H} &= -L\Gamma \end{aligned}$$

we have $e(k+1) = \bar{F}e(k)$. We can easily design the observer by choosing a proper $\bar{F}$. Now let $L = \begin{bmatrix} 0.5 & 0 \end{bmatrix}$, we have the following parameters

$$\begin{aligned}
\bar{F} &= 1 - \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= 0.5 \\
\bar{G} &= 0.5 \begin{bmatrix} 0.5 & 0 \end{bmatrix} - \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0.2 & 0.8 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \\
\bar{H} &= -\begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= -0.5
\end{aligned}$$

We then have the observer dynamic given by

$$z(k+1) = 0.5z(k) + -0.5x_2(k) - 0.5u(k)$$

and the error of the estimation goes to 0 exponentially, or $\hat{v}(k) \to v(k)$ as

$$e(k+1) = 0.5e(k)$$

(c) In this part, since both the state x and disturbance v are not measurable, we may still adopting the controller from Part (a), $u(k) = -Lx(k) - L_v v(k)$, provided we are able to estimate the state x and disturbance v. We modify the controller as follows

$$u(k) = -L\hat{x}(k) - L_v\hat{v}(k)$$

as $\hat{x}$ and $\hat{v}$ are the estimated states respectively, where

$$\begin{aligned}
L &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\
L_v &= 1
\end{aligned}$$

First of all, we check the observability of the augmented system in Part (a),

$$W_o = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 1 \\ 0.45 & 1.3 & 1.5 \end{bmatrix}$$

which is of full-rank, and thus the system is observable. Now let us build a full-order observer

$$\begin{aligned}
\begin{bmatrix} \hat{x}(k+1) \\ \hat{v}(k+1) \end{bmatrix} &= \begin{bmatrix} 0.5 & 1 & 1 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{v}(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(k) + K(y(k) - \hat{y}(k)) \\
\hat{y}(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{v}(k) \end{bmatrix}
\end{aligned}$$

By placing the observer poles at 0, we have $A_o(z) = z^3$. Using Ackermann's formula, we can calculate the observer gain

$$\begin{aligned}
K &= A_o(\Phi_{aug}) W_o^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.485 & 1.490 & 1.950 \\ 0.298 & 0.932 & 0.460 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 7.5 & -5 \\ 1 & -6.5 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2.30 \\ -2.36 \\ 5 \end{bmatrix}
\end{aligned}$$

3. (a) From the reference model, we know $A_m(z) = z^2$. We now find the controllability matrix,

$$W_c = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which is of full-rank and thus the system is controllable. By using Ackermann's formula, we have

$$\begin{aligned} L &= \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} A_m(\Phi) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5 \end{bmatrix} \end{aligned}$$

where $u_{fb}(k) = -Lx(k)$ is the feedback controller.

(b) The feedback system is

$$\begin{aligned} x(k+1) &= \left[\begin{bmatrix} 1 & 5 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 5 \end{bmatrix} \right] x(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} u_{ff}(k) \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} u_{ff}(k) \\ y(k) &= \begin{bmatrix} 1 & 0.5 \end{bmatrix} x(k) \end{aligned}$$

Transfer function of the feedback system is then

$$\begin{aligned} H_{fb}(z) &= \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} z & 0 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{z+0.5}{z^2} \end{aligned}$$

We know $H_{ff}(z)H_{fb}(z) = H_m(z)$, since $z = -0.5$ is a stable zero, we can design

$$H_{ff}(z) = \frac{z-0.5}{z+0.5}$$

to fulfill the requirements.

(c) We first check the observability matrix

$$W_o = \begin{bmatrix} 1 & 0.5 \\ 1.5 & 5 \end{bmatrix}$$

which is of full rank. We then can design an observer, and modify the control law to be

$$u_{fb}(k) = -L\hat{x}(k)$$

The observer poles can be arbitrary assigned by using Ackermann's formula. The feed-forward controller will remain the same since it only needs the input $u_c(k)$. Therefore it is still possible to use the two-degree-of-freedom controller to make the whole system follow the reference model, as long as the observer has stable poles.