EE5103 - Computer Control Systems
Homework #1

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1. (a) Given Kirchhoff's voltage law,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = e_i \quad (1)$$

$$\frac{1}{C}\int idt = e_0 \quad (2)$$

Taking La-place transform of (1) yield

$$\begin{aligned} sLI(s) + RI(s) + \frac{1}{sC}I(s) &= E_i(s) \\ \frac{E_i(s)}{I(s)} &= sL + R + \frac{1}{sC} \end{aligned}$$

while taking La-place transform of (2) yeild

$$\begin{aligned} \frac{1}{sC}I(s) &= E_0(s) \\ \frac{E_0(s)}{I(s)} &= \frac{1}{sC} \end{aligned}$$

Transfer function,

$$\begin{aligned} \frac{E_0(s)}{E_i(s)} = \frac{E_0(s)}{I(s)}\frac{I(s)}{E_i(s)} &= \frac{1}{s^2LC + sRC + 1} \\ &= \frac{2}{s^2 + s + 2} \end{aligned}$$

(b) Let

$$\begin{aligned} x_1 &= e_0 \\ x_2 &= \dot{x}_1 \end{aligned}$$

Then

$$\begin{aligned} \dot{x}_2 &= \ddot{e}_0 \\ &= \frac{1}{C}\frac{di}{dt} \\ &= \frac{1}{CL}\left[e_i - Ri - \frac{1}{C}\int idt\right] \\ &= \frac{1}{CL}\left[e_i - RCx_2 - x_1\right] \end{aligned}$$

Written in state-space form, we have

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{CL} \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

(c) We first calculate

$$\begin{aligned} L\{e^{At}\} &= (sI - A)^{-1} \\ &= \begin{bmatrix} s & -1 \\ 2 & s+1 \end{bmatrix}^{-1} \\ &= \frac{1}{s^2+s+2} \begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix} \\ &= \frac{1}{(s+0.5)^2+1.75} \begin{bmatrix} (s+0.5)+0.5 & 1 \\ -2 & (s+0.5)-0.5 \end{bmatrix} \end{aligned}$$

Taking the inverse La-place transform of it and by referring to the La-place table, we have

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{-0.5t}\cos(\sqrt{1.75}t) + \frac{0.5}{\sqrt{1.75}} e^{-0.5t}\sin(\sqrt{1.75}t) & \frac{1}{\sqrt{1.75}} e^{-0.5t}\sin(\sqrt{1.75}t) \\ -\frac{2}{\sqrt{1.75}} e^{-0.5t}\sin(\sqrt{1.75}t) & e^{-0.5t}\cos(\sqrt{1.75}t) - \frac{0.5}{\sqrt{1.75}} e^{-0.5t}\sin(\sqrt{1.75}t) \end{bmatrix} \\ &= e^{-0.5t} \begin{bmatrix} \cos(1.3229t) + 0.3780\sin(1.3229t) & 0.7559\sin(1.3229t) \\ -1.5119\sin(1.3229t) & \cos(1.3229t) - 0.3780\sin(1.3229t) \end{bmatrix} \end{aligned}$$

Now, to compute e^{Ah} with $h = 1$,

$$\begin{aligned} \Phi &= e^{A} \\ &= e^{At}|_{t=1} \\ &= 0.6065 \begin{bmatrix} 0.6118 & 0.7328 \\ -1.4656 & -0.1211 \end{bmatrix} \\ &= \begin{bmatrix} 0.3711 & 0.4445 \\ -0.889 & -0.0734 \end{bmatrix} \\ \Gamma &= \int_0^1 e^{At} dt \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0.7590 & 0.3145 \\ -6289 & 0.4445 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0.6289 \\ 0.889 \end{bmatrix} \end{aligned}$$

Then, the sampled system is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.3711 & 0.4445 \\ -0.889 & -0.0734 \end{bmatrix} x(k) + \begin{bmatrix} 0.6289 \\ 0.889 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned}$$

(d) By applying z-transform to the system above, we have

$$\begin{aligned} z(X(z) - x(0)) &= \Phi X(z) + \Gamma U(z) \\ X(z) &= (zI - \Phi)^{-1}(zx(0) + \Gamma U(z)) \\ Y(z) &= C(zI - \Phi)^{-1} zx(0) + C(zI - \Phi)^{-1}\Gamma U(z) \end{aligned}$$

Transfer function will be

$$\begin{aligned} H(z) &= C(zI-\Phi)^{-1}\Gamma \\ &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} z-0.3711 & -0.4445 \\ 0.889 & z+0.0734\end{bmatrix}^{-1}\begin{bmatrix}0.6289\\0.889\end{bmatrix} \\ &= \frac{0.6289z+0.4413}{z^2-0.2977z+0.3679} \end{aligned}$$

Input-output model can then be derived by taking the inverse z-transform as follows

$$\begin{aligned} (z^2-0.2977z+0.3679)Y(z) &= (0.6289z+0.4413)U(z) \\ y(k+2)-0.2977y(k+1)+0.3679y(k) &= 0.6289u(k+1)+0.4413u(k), \quad k\geq 0 \end{aligned}$$

(e) Since we know the system, the input signal, and also the initial condition, we can simply substitute them into

$$\begin{aligned} Y(z) &= C(zI-\Phi)^{-1}zx(0)+H(z)U(z) \\ &= z\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} z-0.3711 & -0.4445 \\ 0.889 & z+0.0734\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix} + \frac{0.6289z+0.4413}{z^2-0.2977z+0.3679}\frac{z}{z-1} \\ &= \frac{z(z+0.0734)}{z^2-0.2977z+0.3679} + \frac{z(0.6289z+0.4413)}{(z^2-0.2977z+0.3679)(z-1)} \\ &= \frac{z(z+0.0734)(z-1)+z(0.6289z+0.4413)}{(z^2-0.2977z+0.3679)(z-1)} \\ &= \frac{z(z^2-0.2977z+0.3679)}{(z^2-0.2977z+0.3679)(z-1)} \\ &= \frac{z}{z-1} \end{aligned}$$

In time domain, $y(k)=1$ for all $k\geq 0$.

2. (a) Poles of the system is at $s=-1$, while zeros of the system is at $s=1$. Since the poles are on the left half plane, the system is stable. When the system is inverted, zeros of the system will be the pole of the inverse, and therefore it does not have a stable inverse due to the unstable zero at $s=1$.

(b) Transfer function,

$$G(s)=\frac{s-1}{(s+1)^2}=\frac{s}{(s+1)^2}-\frac{1}{(s+1)^2}$$

Taking z-transform of the transfer function with sampling period h, we have

$$\begin{aligned} G(z) &= \frac{(z-1)he^{-h}}{(z-e^{-h})^2} - \frac{(1-e^{-h}(1+h))z+e^{-h}(e^{-h}+h-1)}{(z-e^{-h})^2} \\ &= \frac{ze^{-h}(2h+1-e^{h})-e^{-h}(2h-1+e^{-h})}{(z-e^{-h})^2} \end{aligned}$$

Pole location,

$$z=e^{-h}<1$$

for all $h>0$. Therefore the sampled system can be made stable.

(c) Similarly for the inverse system, we have

$$G_i(z)=\frac{(z-e^{-h})^2}{ze^{-h}(2h+1-e^{h})-e^{-h}(2h-1+e^{-h})}$$

Poles location,

$$\begin{aligned} ze^{-h}(2h+1-e^{h}) - e^{-h}(2h-1+e^{-h}) &= 0 \\ z &= \frac{2h-1+e^{-h}}{2h+1-e^{h}} \end{aligned}$$

We can simply choose h to be large enough such that $e^h >>> 2h$ and thus $z \to 0$. For example, by assigning $h = 10$, we have $z = -0.000863 < 1$. So it is able to be made stable.

3. (a) Transfer function of the system can be derived as follows,

$$\begin{aligned} C(zI-\Phi)^{-1}\Gamma &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & -2 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{2}{(z-2)(z+1)} \end{aligned}$$

Poles of the system is at $z = 2, -1$ which is unstable. Now we look at the controllable and observable matrices,

$$\begin{aligned} W_c &= \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \\ W_o &= \begin{bmatrix} C \\ C\Phi \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

both are of full rank. Therefore the system is controllable and observable.

(b) As derived in the previous part, the transfer function of the system is

$$\begin{aligned} C(zI-\Phi)^{-1}\Gamma &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & -2 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{2}{(z-2)(z+1)} = \frac{Y(z)}{U(z)} \end{aligned}$$

We then have

$$\begin{aligned} (z^2-z-2)Y(z) &= 2U(z) \\ y(k+2) - y(k+1) - 2y(k) &= 2u(k), \quad k \geq 0 \end{aligned}$$

(c) By taking the z-transform of the controller, we have

$$U(z) = K(U_c(z) - Y(z))$$

Substitute it to the open loop transfer function,

$$\begin{aligned} \frac{Y(z)}{U(z)} &= \frac{2}{z^2-z-2} \\ Y(z) &= \frac{2}{z^2-z-2}U(z) \\ &= \frac{2K}{z^2-z-2}U_c(z) - \frac{2K}{z^2-z-2}Y(z) \\ (z^2-z-2)Y(z) + 2KY(z) &= 2KU_c(z) \end{aligned}$$

Transfer function of the closed-loop system is

$$\frac{Y(z)}{U_c(z)} = \frac{2K}{z^2-z+(2K-2)}$$

(d) We have the characteristic equation given by

$$A(z) = z^2 - z + (2K - 2)$$

We can form the Jury's Criterion table as shown below

$$\begin{array}{llll}
1 & \text{-1} & 2K-2 & \\
2K-2 & \text{-1} & 1 & \alpha_2 = 2K-2 \\
\hline
1-(2K-2)^2 & -1+(2K-2) & & \\
-1+(2K-2) & 1-(2K-2)^2 & & \alpha_1 = \frac{2K-3}{1-(2K-2)^2} \\
\hline
1-(2K-2)^2 - \frac{(2K-3)^2}{1-(2K-2)^2} & & &
\end{array}$$

Now since $a_0 = 1 > 0$, we must have

$$\begin{aligned}
1-(2K-2)^2 &> 0 &(3)\\
1-(2K-2)^2 - \frac{(2K-3)^2}{1-(2K-2)^2} &> 0 &(4)
\end{aligned}$$

in order to have all poles located in the unit disk according to Jury's Stability Criterion. From (3), we can deduce that

$$\begin{aligned}
-4K^2 - 3 + 8K &> 0\\
4K^2 - 8K + 3 &< 0\\
(2K-3)(2K-1) &< 0
\end{aligned}$$

and the solution to the above inequality is just

$$\frac{1}{2} < K < \frac{3}{2} \qquad (5)$$

Similarly from (4), since $1 - (2K-2)^2 > 0$, we have

$$\begin{aligned}
[1-(2K-2)^2]^2 - (2K-3)^2 &> 0\\
(-4K^2 - 3 + 8K)^2 - (2K-3)^2 &> 0\\
(-4K^2 + 8K - 3 - 2K + 3)(-4K^2 + 8K - 3 + 2K - 3) &> 0\\
K(2K-3)^2(K-1) &> 0
\end{aligned}$$

Solution to the above inequality is

$$K < 0 \text{ or } K > 1, \quad K \neq \frac{3}{2} \qquad (6)$$

Finally, combine inequality (5) and (6) to get the final answer

$$1 < K < \frac{3}{2}$$

(e) Suppose the error signal is

$$e(k) = u_c(k) - y(k)$$

Taking the z-transform of it, we have

$$\begin{aligned}
E(z) &= U_c(z) - Y(z)\\
Y(z) &= U_c(z) - E(z)
\end{aligned}$$

Substitute it to the transfer function,

$$\begin{aligned}
\frac{Y(z)}{U_c(z)} &= \frac{2K}{z^2 - z + (2K-2)} \\
\frac{U_c(z) - E(z)}{U_c(z)} &= \frac{2K}{z^2 - z + (2K-2)} \\
E(z) &= \frac{z^2 - z - 2}{z^2 - z + (2K-2)} U_c(z)
\end{aligned}$$

Since u_c is a unit step function,

$$U_c(z) = \frac{z}{z-1}$$

By applying the Final Value Theorem,

$$\begin{aligned}
e(\infty) &= \lim_{z \to 1} (z-1) \frac{z^2 - z - 2}{z^2 - z + (2K-2)} \frac{z}{z-1} \\
&= \frac{-2}{2K-2} \\
&= \frac{1}{1-K}
\end{aligned}$$

4. (a) Taking z-transform of the difference equation, we have

$$\begin{aligned}
zY(z) &= 3Y(z) - 2z^{-1}Y(z) + z^{-1}U(z) - 2z^{-2}U(z) \\
\frac{Y(z)}{U(z)} &= \frac{z^{-1} - 2z^{-2}}{z - 3 + 2z^{-1}} \\
&= \frac{z-2}{z^3 - 3z^2 + 2z}
\end{aligned}$$

Poles location,

$$\begin{aligned}
z^3 - 3z^2 + 2z &= 0 \\
z(z-1)(z-2) &= 0 \\
z &= 0, 1, 2
\end{aligned}$$

which is unstable at $z = 2 > 1$. The inverse system has pole locations at

$$\begin{aligned}
z - 2 &= 0 \\
z &= 2
\end{aligned}$$

which is also unstable.

(b) In general, if there are common poles and zeros in a system, the minimal-realization of the system is observable but not controllable. Therefore we can NOT realize the system such that it is both controllable and observable due to the common pole and zero pair, $z = 2$.

(c) It is possible to realize the system such that it is observable but not controllable. We can simply use the observable canonical realization to realize this system, as follows

$$\begin{aligned}
z(k+1) &= \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(k)
\end{aligned}$$

It can be verified easily that the observability matrix

$$W_o = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix}$$

is of full rank. Thus it is observable. Similarly, we can see that the controllability matrix

$$W_c = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -2 \\ -2 & 0 & 0 \end{bmatrix}$$

has rank 2 (not full rank). Thus the system is uncontrollable.

(d) It is NOT possible to realize the system such that it is controllable but not observable, although one might think that we can simply use the controllable canonical realization to realize this system, as follows

$$\begin{aligned} z(k+1) &= \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x(k) \end{aligned}$$

It can be verified easily that the controllability matrix is of full rank. Thus it seems to be controllable. Similarly, we can see that the observability matrix has rank 2 (not full rank). Thus the system is unobservable. However, if one write down the transfer function of this realization, we have

$$y(k+2) = y(k+1) + u(k)$$

which is NOT a realization for the system. Therefore we can NOT realize the system in the controllable canonical form.

(e) We simply add in a row and a column of zeroes to the observable canonical form, we have

$$\begin{aligned} z(k+1) &= \begin{bmatrix} 3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(k) \end{aligned}$$

We can easily verify that it is a realization of the given difference equation. Now let us find the controllable and observable matrices,

$$\begin{aligned} W_c &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & -2 & -2 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ W_o &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 7 & 3 & 1 & 0 \\ 15 & 7 & 3 & 0 \end{bmatrix} \end{aligned}$$

where both of them are NOT full rank. Therefore this is an uncontrollable and unobservable realization of the given system.