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New Formula for Computing Residues for Higher Order Poles

The second formula of the Calculation of Residues on Page 81 of the EE2012 Presentation Slides can be generalized as the following.

2. Suppose that $f(z)$ has an n-th order pole at $z = z_0$. Then

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right]$$

where $m \geq n$ is an integer.

[Remark by the course lecturer: This is a very creative way of thinking. The main idea for such a generalization proposed by Swee King is that for certain situations, the above procedure might yield a relatively simpler or straightforward computation, which is demonstrated in an example after the following proof. Ben Chen]

Proof. Consider function $f(z)$ which has an n-th order pole at $z = z_0$. The Laurent expansion of this function is given by:

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + \cdots$$

Multiply $f(z)$ by $(z - z_0)^m$ with $m \geq n$

$$(z - z_0)^m f(z) = a_{-n}(z - z_0)^{m-n} + a_{-(n-1)}(z - z_0)^{m-(n-1)} + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + \cdots$$

We have

$$\lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right]$$
$$= \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[a_{-n}(z - z_0)^{m-n} + a_{-(n-1)}(z - z_0)^{m-(n-1)} + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots \right] \right]$$

Note that every term above is in the form of

$$\lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[a_{-N}(z - z_0)^{m-N} \right] \right]$$

For the terms with $N > 1$, say $N = p + 1$ with $p > 0$,

$$\lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[a_{-N}(z - z_0)^{m-N} \right] \right] = \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[a_{-N}(z - z_0)^{(m-1)-p} \right] \right] = 0$$

Next, for the case, in which $N < 1$,

$$\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left[a_{-N}(z-z_0)^{m-N}\right]\right] = \lim_{z \to z_0}\left[\frac{(m-N)!}{(1-N)!}\left[a_{-N}(z-z_0)^{1-N}\right]\right] = 0$$

Finally, the term with $N = 1$ has

$$\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left[a_{-1}(z-z_0)^{m-1}\right]\right] = a_{-1}(m-1)!$$

Hence,

$$\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right] = a_{-1}(m-1)! \quad \Rightarrow \quad a_{-1} = \frac{1}{(m-1)!}\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right]$$

By the definition of the residue,

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{(m-1)!}\lim_{z \to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right]$$

where $m \geq n$, n = Order of pole of $f(z)$ at $z = z_0$ *QED*

Example: The extra freedom in selecting *m* in the proposed formula can simplify some problems in computing residues and thus complex integrals. Consider Example (a) on Page 63 of Presentation Slides, i.e.,

$$\oint_{|z|=1} \frac{\sin z}{z^4}\, dz$$

It can be easily shown that it has a 3rd order pole at $z_0 = 0$. However, if we use *m* = 4, we will find that it is much easier to compute the residue compared with that using the original formula with $m = 3$, i.e.,

$$\oint_{|z|=1} \frac{\sin z}{z^4}\, dz = 2\pi i \frac{1}{3!}\lim_{z \to 0}\left[\frac{d^3}{dz^3}\left(z^4 \cdot \frac{\sin z}{z^4}\right)\right] = 2\pi i \frac{1}{3!}\lim_{z \to 0}\left[\frac{d^3}{dz^3}(\sin z)\right] = 2\pi i \frac{1}{3!}(-\cos 0) = -\frac{\pi i}{3}$$

[Well done**!** The annex is the procedure using the original formula. Ben Chen]

Annex: Using Formula 2 on page 81 of the presentation slides, we have

$$
\begin{aligned}
\mathrm{Res}(f,0) &= \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[z^3\frac{\sin z}{z^4}\right] = \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{\sin z}{z}\right] \\
&= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[\frac{z\cos z - \sin z}{z^2}\right] \\
&= \frac{1}{2}\lim_{z\to 0}\frac{2\sin z - z^2\sin z - 2z\cos z}{z^3} \\
&= \frac{1}{2}\lim_{z\to 0}\frac{(2\sin z - z^2\sin z - 2z\cos z)'}{(z^3)'} \qquad \text{(L'Hospital Rule)} \\
&= \frac{1}{2}\lim_{z\to 0}\frac{-z^2\cos z}{3z^2} \\
&= -\frac{1}{6}
\end{aligned}
$$

Thus,

$$
\oint_{|z|=1}\frac{\sin z}{z^4}\,dz = 2\pi i\cdot\mathrm{Res}(f,0) = -\frac{\pi i}{3}
$$

The advantage of using the formula proposed by Swee King is obvious. Ben Chen